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Locating the peaks of the least energy solutions to an ellyptic system with Neumann boundary conditions ☆

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Abstract

We consider a system of the form $-\varepsilon^2\Delta u + u = g(v)$, $-\varepsilon^2\Delta v + v = f(u)$ in Ω with Neumann boundary condition on $\partial\Omega$, where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geqslant 3$ and f, g are power-type nonlinearities having superlinear and subcritical growth at infinity. We prove that the least energy solutions to such a system concentrate, as ε goes to zero, at a point of the boundary which maximizes the mean curvature of the boundary of Ω.

Keywords: Superlinear elliptic systems; Spike-layered solutions; Positive solutions; Minimax methods

1. Introduction

We are concerned with the following system of elliptic equations with Neumann boundary conditions:

$$-\varepsilon^2\Delta u + u = g(v) \quad \text{in } \Omega, \tag{1.1}$$

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$$-\varepsilon^2 \Delta v + v = f(u) \quad \text{in } \Omega, \tag{1.2}$$

$$u, v > 0 \quad \text{in } \Omega, \tag{1.3}$$

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \quad \text{on } \partial\Omega, \tag{1.4}$$

where $\varepsilon > 0$ is a small parameter, Ω is a C^2 bounded domain of $\mathbb{R}^N$, with $N \geqslant 3$.

We assume

(H) $f, g \in C^1(\mathbb{R})$, $f(0) = 0 = f'(0)$, $g(0) = 0 = g'(0)$ and there exist real numbers ℓ_1, $\ell_2 > 0$ and $p, q > 2$ such that $\frac{1}{p} + \frac{1}{q} > \frac{N-2}{N}$ and

$$\lim_{|s| \to \infty} \frac{f'(s)}{|s|^{p-2}} = \ell_1, \qquad \lim_{|s| \to \infty} \frac{g'(s)}{|s|^{q-2}} = \ell_2. \tag{1.5}$$

Moreover, for some $\delta > 0$ and every $s \in \mathbb{R}$, $s \neq 0$,

$$f(s)s \geqslant (2+\delta)F(s) > 0 \quad \text{and} \quad f^2(s) \leqslant 2f'(s)F(s) \tag{1.6}$$

and

$$g(s)s \geqslant (2+\delta)G(s) > 0 \quad \text{and} \quad g^2(s) \leqslant 2g'(s)G(s), \tag{1.7}$$

where $F(s) := \int_0^s f(\sigma)\, d\sigma$ and $G(s) := \int_0^s g(\sigma)\, d\sigma$. Since we look for positive solutions, we let $f(s) = g(s) = 0$ for $s \leqslant 0$.

For example, we can consider nonlinearities such as

$$f(s) = A(s^+)^a + (s^+)^{p-1}, \quad g(s) = B(s^+)^b + (s^+)^{q-1},$$

$$A, B \geqslant 0,\ a \in (2, p-1],\ b \in (2, q-1], \quad \frac{1}{p} + \frac{1}{q} > \frac{N-2}{N}. \tag{1.8}$$

Let us recall some well-known results about the equation

$$-\varepsilon^2 \Delta u + u = u^p \quad \text{in } \Omega, \tag{1.9}$$

$$u > 0 \quad \text{in } \Omega, \tag{1.10}$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega, \tag{1.11}$$

where $\varepsilon > 0$ is a small parameter, Ω is a smooth bounded domain of $\mathbb{R}^N$, with $N \geqslant 3$ and $1 \leqslant p < \frac{2N}{N-2}$. In [5–7], the authors proved the existence of a nontrivial solution u_ε

to problem (1.9)–(1.11) for ε small enough, they showed that u_ε attains its maximum value at a point $P_\varepsilon \in \partial\Omega$ and that, up to subsequences, P_ε, as ε goes to zero, approaches a point P_0, which maximizes the mean curvature of the boundary of Ω.

In [2], the authors consider system (1.1)–(1.4) with special nonlinearities given in (1.8) with $A = B = 0$ and, using a dual variational formulation of the problem, they proved that there exist nontrivial positive solutions u_ε and v_ε provided ε is small enough. Moreover, if $p, q < \frac{2N}{N-2}$, u_ε and v_ε have global maximum point at points P_ε and Q_ε, respectively, which belong to the boundary of Ω and have a common limit point.

In [8], Ramos and Yang, using a variational argument, consider a more general class of nonlinearities and they prove the following result.

Theorem 1.1. *Under assumptions* (H), *there exists* $\varepsilon_0 > 0$ *such that for any* $0 < \varepsilon < \varepsilon_0$ *problem* (1.1)–(4.1) *has nonconstant positive solutions* $u_\varepsilon, v_\varepsilon \in C^2(\bar{\Omega})$. *Moreover, both functions* u_ε *and* v_ε *attain their maximum value at some unique and common point* $P_\varepsilon \in \partial\Omega$.

At this point a natural question arises: *if* P_ε approaches a point P_0 as ε goes to zero, is it true that P_0 maximizes the mean curvature of the boundary of Ω? The answer is known to be positive for the single equation (cf. [4,7]) and for the particular case of the system treated in [2]. In the present paper we extend these results to our more general framework.

Theorem 1.2. *Let, up to a subsequence,* $P_0 := \lim_{\varepsilon \to 0} P_\varepsilon$ (*see Theorem* 1.1). *Then*

$$H(P_0) = \max_{P \in \partial\Omega} H(P),$$

where $H(P)$ *denotes the mean curvature of* $\partial\Omega$ *at the boundary point* P.

Here we use some ideas introduced by Del Pino and Felmer [4], where they essentially proved the same result of Ni and Takagi (i.e. the least energy solutions u_ε to (1.9)–(1.11) concentrate around the maximum point of the mean curvature of the boundary, as ε goes to zero), but using a different method. More precisely we recall that, in order to prove such a result, Ni and Takagi estimate the error commited when approximating the scaled function $\widetilde{u}_\varepsilon(x) := u_\varepsilon(\varepsilon x + P_\varepsilon)$ by its limit u which solves the limiting equation

$$-\Delta u + u = u^p \quad \text{in } \mathbb{R}^N, \tag{1.12}$$

$$u > 0 \quad \text{in } \mathbb{R}^N, \quad u(0) = \max_{x \in \mathbb{R}^N} u(x), \tag{1.13}$$

$$\lim_{|x| \to +\infty} u(x) = 0. \tag{1.14}$$

At this aim they need the crucial assumption that solution to (1.12)–(1.14) is *unique*. On the other hand, Del Pino and Felmer [4] do not need this condition, because they estimate the error commited when approximating the energy of the least energy solution u_ε to (1.9)–(1.11) by the energy of the least energy solution of the limiting problem (1.12)–(1.14).

In our case we point out that a uniqueness result for the limiting system

$$-\Delta u + u = g(v) \quad \text{in } \mathbb{R}^N, \tag{1.15}$$

$$-\Delta v + v = f(u) \quad \text{in } \mathbb{R}^N, \tag{1.16}$$

$$u, v > 0 \quad \text{in } \mathbb{R}^N \tag{1.17}$$

is not known. On the other hand, if we denote by I is the (strongly indefinite) energy functional associated to the system, namely

$$I(u,v) = \int_\Omega (\varepsilon^2 \langle \nabla u, \nabla v \rangle + uv - F(u) - G(v))\, dx, \tag{1.18}$$

then it is known that the underlying minimax theorem associated to its ground-state level is an infinite-dimensional linking theorem; this is in contrast with the scalar case (single equation), whose corresponding ground-state level is a one-dimensional mountain-pass energy level. Nevertheless, we are able to fully exploit a variational characterization (of Nehari type) used in [8] so as to derive sharp estimates on the ground-state levels $I(u_\varepsilon, v_\varepsilon)$, where $(u_\varepsilon, v_\varepsilon)$ are the solutions to (1.1)–(1.4) given in [8] (see Theorems 3.4 and 4.2).

The paper is organized as follows. In Section 2, we recall some results obtained in [8]. In Section 3, we estimate the ground-state level from below and in Section 4, we estimate the ground-state level from above.

2. Some known results

Let $(u_\varepsilon, v_\varepsilon)$ be any ground-state solution to (1.1)–(1.4), that is, $I(u_\varepsilon, v_\varepsilon)$ is the least critical level among all nonzero solutions of the problem. Moreover, we fix $(u_\varepsilon, v_\varepsilon)$ in such a way that their Morse indices is not greater than 2 (see [8] for a discussion of this point).

Let x_ε be the common maximum point of u_ε and v_ε (see Theorem 1.1). We fix a sequence $\varepsilon_j \to 0$ in such a way that $\widetilde{x}_j := x_{\varepsilon_j} \to P_0 \in \partial\Omega$ and consider the corresponding rescaled solutions of the system:

$$\widetilde{u}_j(x) = u_{\varepsilon_j}(\varepsilon_j x + \widetilde{x}_j), \quad \widetilde{v}_j(x) = v_{\varepsilon_j}(\varepsilon_j x + \widetilde{x}_j), \quad x \in \widetilde{\Omega}_j := \frac{\Omega - \widetilde{x}_j}{\varepsilon_j}.$$

According to Theorem 1.1 in [8], in the following we may assume without loss of generality that $2<q=p<2N/(N-2)$. Also, in order to simplify the notations, we assume that the unit outward normal of Ω at the point P_0 is the vector $(0,\dots,0,-1)\in\mathbb{R}^N$.

Next we recall some facts proved in [8].

Proposition 2.1. *As $j\to\infty$, $\tilde{u}_j$ and $\tilde{v}_j$ converge in C^2_{loc} to positive solutions $u,v\in H^2(\mathbb{R}^N_+)\cap C^2(\mathbb{R}^N_+)$ of the system*

$$-\Delta u+u=g(v)\quad\text{in }\mathbb{R}^N,\tag{2.1}$$

$$-\Delta v+v=f(u)\quad\text{in }\mathbb{R}^N,\tag{2.2}$$

$$u,v>0\quad\text{in }\mathbb{R}^N,\tag{2.3}$$

$$\frac{\partial u}{\partial n}=\frac{\partial v}{\partial n}=0\quad\text{on }\partial\mathbb{R}^N_+,\tag{2.4}$$

where $\mathbb{R}^N_+=\{x\in\mathbb{R}^N : x_N>0\}$.

It is known (cf. [3]) that u,v can be seen as restrictions to $\mathbb{R}^N_+$ of radially symmetric solutions of the system in the whole space $\mathbb{R}^N$.

Lemma 2.2. *For any $\delta>0$ there exists $j_0\in In$ and $R>0$ such that, for every $j\geqslant j_0$,*

$$\int_{\tilde{\Omega}_j\cap\{x:\,|x|\geqslant R\}}(|\nabla\tilde{u}_j^2+|\nabla\tilde{v}_j|^2+\tilde{u}_j^2+\tilde{v}_j^2+f(\tilde{u}_j)\tilde{u}_j+g(\tilde{v}_j)\tilde{v}_j)\leqslant\delta,\tag{2.5}$$

and also

$$\int_{\tilde{\Omega}_j\cap\{x:\,|x|\geqslant R\}}((f\tilde{u}_j)\tilde{u}_j+g(\tilde{v}_j)\tilde{v}_j)\,|x_N|)\leqslant\delta.$$

Moreover, there exists $C>0$ such that

$$\int_{\tilde{\Omega}_j}(|D^2\tilde{u}_j|^2+|D^2\tilde{u}_j|^2)\leqslant C,\quad\forall\, j\in\mathrm{IN}.$$

Proof. The first conclusion in the Lemma was proved in [8, Proposition 1.6], by using the information on the Morse index of the solutions. It also proved in [8, Theorem 2.1] that this leads to the conclusion that the solutions decay uniformly in j, i.e. for any given $\delta>0$ we can find $R>0$ and $j_0\in\mathbb{N}$ such that

$$\tilde{u}_j(x)+\tilde{v}_j(x)+|\nabla\tilde{u}_j(x)|+|\nabla\tilde{v}_j(x)|\leqslant\delta,\qquad\forall j\geqslant j_0,\ x\in\tilde{\Omega}_j:|x|\geqslant R.$$

Now, fix any $0<a<1$ and let $\Psi(x)=e^{a|x|}$. Let φ be a cut-off function such that $\varphi=0$ in $B_{R/2}(0)$ and $\varphi=1$ in $\mathbb{R}^N\backslash B_R(0))$, where $R=R(\delta)$.

By choosing a small $0<\delta<1-a$, we multiply our equations by $\widetilde{u}_j\Psi^2\varphi^2$ and $\widetilde{v}_j\Psi^2\varphi^2$ respectively, we integrate by parts and use straightforward computations to deduce, thanks to (2.5) and the fact that $f(s)/s\to 0$ and $g(s)/s\to 0$ as $s\to 0$, that, for some $C>0$,

$$\int_{\widetilde{\Omega}_j\cap\{x:\ |x|\geqslant R\}}((f(\widetilde{u}_j)\widetilde{u}_j+g((\widetilde{v}_j)\widetilde{v}_j)\ \Psi)\leqslant C,$$

thus also, for some C',

$$\int_{\widetilde{\Omega}_j}((f(\widetilde{u}_j)\widetilde{u}_j+g(\widetilde{v}_j)\widetilde{v}_j\ \Psi)\leqslant C'.$$

This implies the second conclusion in the Lemma. As for the boundedness of the second derivatives, we simply observe that since $\frac{\partial\widetilde{u}_j}{\partial n}=0$ and also $\dfrac{\partial}{\partial x_i}\left(\dfrac{\partial\widetilde{u}_j}{\partial n}\right)=0$ for every $j\in\mathbb{N}$ and every $i=1,\dots,N$, then, by integrating by parts twice in $\widetilde{\Omega}_j$,

$$\int_{\widetilde{\Omega}_j}|D^2\widetilde{u}_j|^2=\int_{\widetilde{\Omega}_j}(\Delta\widetilde{u}_j)^2,$$

and similarly for $D^2\widetilde{v}_j$, and the conclusion follows. □

Now let us make some remarks about the mean curvature. Let $P\in\partial\Omega$ and consider a diffeomorphism Φ which straightens a boundary portion near P, as described e.g. in [6, p. 823]. In particular, we assume (through translation and rotation of the coordinate system) that the unit outward normal of Ω at the point P is the vector $(0,\dots,0,-1)\in\mathbb{R}^N$. So, $\Phi(0)=P$, $D\Phi(0)=Id$ and we denote by Ψ the local inverse of Φ. We recall from [6, Lemma A.1], that there exists some constant $C=C(P)$ such that

$$\det D(\Phi(x))=1-Cx_N+\mathrm{O}(|x|^2)\quad\text{as } |x|\to 0. \tag{2.6}$$

By definition, the *mean curvature* of $\partial\Omega$ at the point P is $H(P)=C(P)/(N-1)$. Thus, for example, if $\Omega=\{x_N>\rho(x')\}$ and $\partial\Omega=\{x_N=\rho(x')\}$ where $\rho:\mathbb{R}^{N-1}\to\mathbb{R}$ is such that $P=(P',\rho(0))$ and $\nabla\rho(0)=0$ then, as shown in [6],

$$C(P)=\Delta\rho(0).$$

We observe that in this case $\Phi(x',x_N)=(x',\rho(x'))-x_Nn((x',\rho(x'))$ where $n(y)$ stands for the unit outward normal of Ω at the point $y\in\partial\Omega$; namely, $n((x',\rho(x'))=$

$(\nabla\rho(x'), -1)$ so that

$$\Phi(x', x_N) = (x' - x_N\,\nabla\rho(x'), \rho(x') + x_N)$$

and

$$C(P) = \Delta\rho(0) = \Delta\Phi_N(0).$$

Taking derivatives in the identity $\Psi_N(\Phi(x)) = x_N$ and using the fact that $D\Phi(0) = D\Psi(P) = Id$ we also see that

$$\Delta\Psi_N(P) = -\Delta\Phi_N(0) = -\Delta\rho(0). \tag{2.7}$$

3. Estimating the ground-state levels from above

Given any $P \in \partial\Omega$, let $\Omega_j := \frac{1}{\varepsilon_j}(\Omega - P)$ and

$$u_j(x) = u\left(\frac{\Psi(\varepsilon_j x + P)}{\varepsilon_j}\right)\chi(\Psi(\varepsilon_j x + P)) \in H^1(\Omega_j),$$

where $\chi \in C^\infty(\mathbb{R}^N)$ is radially symmetric (in particular, $\frac{\partial\chi}{\partial x_N} = 0$ on $\partial\mathbb{R}^N_+$), $\chi = 1$ in $B_r(0)$ and $\chi = 0$ in $\mathbb{R}^N \backslash B_{2r}(0)$, for a small $r > 0$. By construction, $\frac{\partial u_j}{\partial n}(x) = 0$ for every $x \in \partial\Omega_j$. We define v_j in a similar way.

In fact, for simplicity of notations we take $P = 0$, so that $\Omega_j := \frac{1}{\varepsilon_j}\Omega$ and

$$u_j(x) = u\left(\frac{\Psi(\varepsilon_j x)}{\varepsilon_j}\right)\chi(\Psi(\varepsilon_j x)) \in H^1(\Omega_j). \tag{3.1}$$

It was observed in [8] that u_j and v_j satisfy, over Ω_j,

$$-\Delta u_j + u_j = g(v_j) + \mu_j(x), \qquad -\Delta v_j + v_j = f(u_j) + v_j(x), \tag{3.2}$$

where $\mu_j(x)$ and $v_j(x)$ are $o(1)$, in the sense that $\int_{\Omega_j} |\mu_j(x)\phi_j(x)|\,dx = o(1)$ as $j \to \infty$ and similarly to $v_j(x)$, provided $||\phi_j||$ is bounded. In fact, a direct computation (see the proof of Lemma 3.1) shows that

$$\int_{\Omega_j} \mu_j(x)\phi_j(x)\,dx = ||\phi_j||\;\; O(\varepsilon_j) \quad \text{as } j \to \infty \tag{3.3}$$

and similarly for $v_j(x)$. We need a more precise estimate.

Lemma 3.1.

$$\int_{\Omega_j} (\mu_j v_j + v_j u_j) = -\varepsilon_j \Delta\rho(0) \int_{\partial\mathbb{R}^N_+} uv + o(\varepsilon_j) \quad as\ j \to \infty.$$

Proof. In computing Δu_j, all terms involving derivatives of χ are affected by a coefficient ε_j and so they are $o(\varepsilon_j)$, since $u, v \in H^1(\mathbb{R}^N_+)$. The remaining terms are of the form

$$\chi \sum_{i,k,\ell} \frac{\partial^2 u}{\partial x_i \partial x_\ell} \frac{\partial \Psi_\ell}{\partial x_k} \frac{\partial \Psi_i}{\partial x_k} + \varepsilon_j \chi \sum_i \frac{\partial u}{\partial x_i} \Delta \Psi_i \tag{3.4}$$

evaluated at appropriate points. Divide the second term by ε_j, multiply it by v_j and proceed similarly by interchanging u_j with v_j, to arrive at the expression

$$\sum_{i=1}^N \int_{\Omega_j} \frac{\partial w}{\partial x_i}\left(\frac{\Psi(\varepsilon_j x)}{\varepsilon_j}\right) \Delta\Psi_i(\varepsilon_j x)\chi^2(\Psi(\varepsilon_j x))\, dx,$$

where $w := uv$. Changing variables through $y = \Psi(\varepsilon_j x)/\varepsilon_j$, this reads as

$$\sum_{i=1}^N \int_{\Psi(\Omega)/\varepsilon_j} \frac{\partial w}{\partial x_i}(y) \Delta\Psi_i(\Phi(\varepsilon_j y))\chi^2(\varepsilon_j y) \det D\Phi(\varepsilon_j y)\, dy.$$

Taking limits and using Lebesgue's dominated convergence theorem we arrive at

$$\sum_{i=1}^N \Delta\Psi_i(0) \int_{\mathbb{R}^N_+} \frac{\partial w}{\partial x_i}.$$

Since $\int_{\mathbb{R}^{N-1}} \nabla_{x'} w(x', x_N)\, dx' = 0$ (as $w(\cdot, x_N)$ is an even function in $\mathbb{R}^{N-1}$), the above expression reduces to $\Delta\Psi_N(0) \int_{\mathbb{R}^N_+} \frac{\partial w}{\partial x_N}$, i.e. $-\Delta\Psi_N(0) \int_{\partial\mathbb{R}^N_+} uv$. Using (2.7) we conclude that

$$\sum_{i=1}^N \int_{\Omega_j} \frac{\partial w}{\partial x_i}\left(\frac{\Psi(\varepsilon_j x)}{\varepsilon_j}\right) \Delta\Psi_i(\varepsilon_j x)\chi^2(\Psi(\varepsilon_j x))\, dx \to \Delta\rho(0) \int_{\partial\mathbb{R}^N_+} uv. \tag{3.5}$$

This is the boundary term which appears in the expression of the statement of Lemma 3.1.

Next we look at the first term in (3.4). Since $D\Psi(\Phi(\varepsilon_j x)) \to D\Psi(0) = Id$, we have that $\frac{\partial \Psi_\ell}{\partial x_k} \frac{\partial \Psi_i}{\partial x_k} = \delta_{i\ell k} o(\varepsilon_j)$ ($\delta_{i\ell k} = 1$ if $i = \ell = j$ and $\delta_{i\ell k} = 0$ otherwise). Thus, using Lebesgue's dominated convergence theorem and the fact that $u, v \in H^2(\mathbb{R}^N_+)$ and satisfy (2.1)–(2.4), we are reduced to the analysis of terms of the form

$$f\left(u\left(\frac{\Psi(\varepsilon_j x)}{\varepsilon_j}\right)\right)\chi(\Psi(\varepsilon_j x)) - f(u_j(x)).$$

Multiply this by any $\phi_j \in H^1(\Omega_j)$ (with $\|\phi_j\|$ bounded) and use the same change of variable as before, to arrive at

$$\int_{\Psi(\Omega)/\varepsilon_j} [f(u(y))\chi(\varepsilon_j y) - f(u(y)\chi(\varepsilon_j y))]\widetilde{\phi}_j(y) \det D\Phi(\varepsilon_j y)\, dy, \tag{3.6}$$

where $\widetilde{\phi}_j(y) := \phi_j(\Phi(\varepsilon_j y)/\varepsilon_j)$ remains bounded in $H^1(\Psi(\Omega)/\varepsilon_j)$. Observe that the term in brackets can be written as

$$f(u(y))\ (\chi(\varepsilon_j y) - 1) + (f(u(y)) - f(u(y)\chi(\varepsilon_j y)));$$

then, since $|f'(u(y))| \leqslant C$, $|f(u(y))| \leqslant Cu(y)$ and $u \in H^1(\mathbb{R}^N_+)$, we see that the quantity in (3.6) is $o(\varepsilon_j)$. $\square$

We denote by I_j (resp. I_∞) the energy functional obtained from (1.18) by setting $\varepsilon = 1$ and by replacing Ω by Ω_j (resp. by replacing Ω by $\mathbb{R}^N_+$).

Lemma 3.2. *We have that*

$$I_j(u_j, v_j) = I_\infty(u, v) - \varepsilon_j(N-1)H(P)\gamma + o(\varepsilon_j) \quad \text{as } j \to \infty,$$

where γ is the positive constant given by

$$\gamma := \gamma(f) + \gamma(g) + \frac{1}{2}\int_{\partial\mathbb{R}^N_+} uv$$

with

$$\gamma(f) := \int_{\mathbb{R}^N_+} \left(\frac{1}{2}f(u)u - F(u)\right) x_N\, dx$$

and similarly for $\gamma(g)$.

Proof. Thanks to (2.5), it follows easily, as in [6, p. 828], that

$$\int_{\Omega_j}\left(\frac{1}{2}f(u_j)u_j - F(u_j)\right) = \int_{\mathbb{R}^N_+}\left(\frac{1}{2}f(u)u - F(u)\right) - \varepsilon_j \Delta\rho(0)\gamma(f) + o(\varepsilon_j).$$

Using this together with (3.2) and Lemma 3.1 yield that

$$\begin{aligned}
I_j(u_j, v_j) &:= \int_{\Omega_j} (\langle \nabla u_j, \nabla v_j \rangle + u_j v_j - F(u_j) - G(v_j)) \\
&= \int_{\Omega_j} \left(\frac{1}{2} f(u_j) u_j - F(u_j) \right) + \int_{\Omega_j} \left(\frac{1}{2} g(v_j) v_j - G(v_j) \right) \\
&\quad + \frac{1}{2} \int_{\Omega_j} (\mu_j v_j + v_j u_j) \\
&= \int_{\mathbb{R}^N_+} \left(\frac{1}{2} f(u) u - F(u) \right) + \int_{\mathbb{R}^N_+} \left(\frac{1}{2} g(v) v - G(v) \right) \\
&\quad - \varepsilon_j \Delta\rho(0)(\gamma(f) + \gamma(g)) + o(\varepsilon_j) + \frac{1}{2} \int_{\Omega_j} (\mu_j v_j + v_j u_j) \\
&= \int_{\mathbb{R}^N_+} \left(\frac{1}{2} f(u) u - F(u) \right) + \int_{\mathbb{R}^N_+} \left(\frac{1}{2} g(v) v - G(v) \right) \\
&\quad - \varepsilon_j \Delta\rho(0) \left(\gamma(f) + \gamma(g) + \frac{1}{2} \int_{\partial \mathbb{R}^N_+} uv \right) + o(\varepsilon_j) \\
&= I_\infty(u, v) - \varepsilon_j \Delta\rho(0)\gamma + o(\varepsilon_j),
\end{aligned}$$

as claimed. □

It is proved in [8, Theorem 3.5] that

$$\sup_{E^- \oplus \mathbb{R}^+(u_j, v_j)} I_j = I_j(u_j, v_j) + o(1), \tag{3.7}$$

where $E^- := \{(\phi, -\phi), \phi \in H^1(\Omega_j)\}$. We need a more precise estimate.

Lemma 3.3. *We have that*

$$\sup_{E^- \oplus \mathbb{R}^+(u_j, v_j)} I_j = I_j(u_j, v_j) + o(\varepsilon_j) \quad \text{as } j \to \infty.$$

Proof. This is somehow proved in [8], in the basis of an argument by contradiction. Since this lemma is a delicate and crucial point in the proof of our main result, we show with some detail how the argument used there can be adapted to our purposes. The proof uses (3.2), (3.3), Lemma 3.1 and the decay property of the solutions (cf. (2.5)). Let the supremum be attained at $s_j(u_j, v_j) + t_j(\phi_j, -\phi_j)$ with $||\phi_j|| = 1$ and $|t_j| + |s_j| \leqslant C$ (it is proved in [8] that this is indeed the case). Denote

$$\chi_j(s, t) := I_j(s(u_j, v_j) + t(\phi_j, -\phi_j)), \tag{3.8}$$

so that (s_j, t_j) is a maximum point for χ_j.

Step 1: We claim that

$$\lim_{j\to\infty} s_j = 1. \tag{3.9}$$

We argue by contradiction, assuming that $|s_j - 1| \geqslant \delta > 0$ along a subsequence. In this case, the functions

$$\psi_j := \frac{t_j}{1 - s_j}\phi_j \tag{3.10}$$

are bounded independently of j. We then consider the new function

$$\alpha_j(s) := I_j(s(u_j, v_j) + (1 - s)(\psi_j, -\psi_j)),$$

which has a maximum point at $s = s_j$. We also denote by $\beta_j(s)$ the function α_j evaluated at points where α_j' vanishes (no implicit function theorem is used here; in fact, an explicit expression for β_j can be derived). So, $\beta_j(s_j) = \alpha_j(s_j)$ and it is proved in [8] that if $\liminf |s_j - 1| > 0$ then there exist $\rho > 0$ and some point $\widetilde{s_j}$ lying between s_j and 1 such that

$$\beta_j(\widetilde{s_j}) < \alpha_j(1) - \rho;$$

moreover, $\beta_j(0) \leqslant 0,\ \beta_j(1) = \alpha_j(1) + o(1)$, $\beta_j'(1) > 0$ for large values of j and $\beta_j'(s) \neq 0$ at any point s such that $\beta_j(s) > 0$. We arrive at a contradiction in each of the following two cases (i) and (ii).

(i) Suppose that $0 < s_j < 1$. Then $\beta_j(0) \leqslant 0 < \alpha_j(1) \leqslant \alpha_j(s_j) = \beta_j(s_j)$ while $\beta_j(\widetilde{s_j}) < \alpha_j(1) - \rho \leqslant \alpha_j(s_j) - \rho = \beta_j(s_j) - \rho < \beta_j(s_j)$, that is

$$\max\{\beta_j(0), \beta_j(\widetilde{s_j})\} < \beta_j(s_j)$$

with $s_j \in]0, \widetilde{s_j}[$. This implies that β_j' must vanish at some point s with $\beta_j(s) > 0$, which is impossible, as we mentioned above.

(ii) Suppose that $1 < s_j$. Then also $\widetilde{s_j} > 1$ and, for large values of j,

$$\beta_j(\widetilde{t_j}) < \alpha_j(1) - \rho < \beta_j(1).$$

Since $\beta_j'(1) > 0$, again β' must vanish at some point s such that $\beta_j(s) > 0$, a contradiction. This establishes (3.9).

Step 2: We claim that there exists $C > 0$ such that

$$|t_j| \leqslant C(|s_j - 1| + \varepsilon_j). \tag{3.11}$$

Denote $\theta_j(t) = \chi_j(s_j, t)$, i.e.

$$\theta_j(t) = I_j(s_j(u_j, v_j) + t(\phi_j, -\phi_j)),$$

so that $\theta_j'(t_j) = 0$. A direct computation shows that $\theta_j''(t) \leqslant -2$ for every t and every j, and moreover

$$|\theta_j'(0)| \leqslant C(|s_j - 1| + \varepsilon_j). \tag{3.12}$$

Concerning (3.12), we observe that

$$\begin{aligned}\theta_j'(0) = (s_j - 1) \int f(u_j)\phi_j - (s_j - 1) \int g(v_j)\phi_j \\ + \int (f(u_j) - f(s_j u_j))\phi_j - \int (g(v_j) - g(s_j v_j))\phi_j \\ + s_j \int (v_j \phi_j - \mu_j \phi_j).\end{aligned}$$

The first two integrals are $O(|s_j - 1|)$ while, according to (3.3) the last integral is $O(\varepsilon_j)$. Concerning the third integral, we observe that $|f'(s)| \leqslant C(|s| + |s|^{p-2})$, thus also $|f'(s)| \leqslant C(|s| + |s|^{p-1})$ for every $s \in \mathbb{R}$ (with $2 < p < 2^\star$), and then Hölder's inequality shows that the integral is $O(|s_j - 1|)$. Similarly for the fourth integral, and this establishes (3.12). Recalling that $\theta_j''(t) \leqslant -2$ for every t and that $\theta_j'(t_j) = 0$, the claim follows from (3.12).

Step 3. We claim that there exists $C > 0$ such that, for every j,

$$|s_j - 1| + |t_j| \leqslant C\varepsilon_j. \tag{3.13}$$

In view of (3.11), we may already assume that $\varepsilon_j \leqslant |s_j - 1|$ for every j. In this case, the functions ψ_j as defined in (3.10) are bounded independently of j and so we consider again $\alpha_j(s)$ as in Step 1. A direct computation and (3.3) show that

$$\alpha_j'(1) = \int \mu_j(v_j + \psi_j) + \int v_j(u_j - \psi_j) = O(\varepsilon_j). \tag{3.14}$$

On the other hand, it was proved in [8] that $\sup_{|s-1|<\delta} \alpha_j''(s) < 0$ if δ is sufficiently small. Since $s_j \to 1$, we conclude that

$$|\alpha_j''(s)| \geqslant \rho > 0 \quad \text{for every } s \text{ betweeen } s_j \text{ and } 1. \tag{3.15}$$

Combining (3.14) and (3.15) yields that $|s_j - 1| \leqslant C\varepsilon_j$. Taking also (3.11) into account, the claim follows.

Step 4: Let χ_j be as in (3.8). Then

$$\frac{\partial \chi_j}{\partial s}(1, 0) = \langle \nabla I_j(u_j, v_j), (u_j, v_j) \rangle = \int (v_j u_j + \mu_j v_j) = o(1)$$

and

$$\frac{\partial \chi_j}{\partial t}(1,0) = \langle \nabla I_j(u_j, v_j), (\phi_j, -\phi_j) \rangle = \int (v_j\phi_j - \mu_j\phi_j) = o(1).$$

Thus

$$\begin{aligned}\chi_j(s_j, t_j) &= \chi_j(1,0) + (s_j - 1)\frac{\partial \chi_j}{\partial s}(1,0) + t_j \frac{\partial \chi_j}{\partial t}(1,0) + o(|s_j - 1| + |t_j|)\\ &= \chi_j(1,0) + o(\varepsilon_j)\end{aligned}$$

as follows from (3.13). □

Finally, consider the ground-state levels $I(u_\varepsilon, v_\varepsilon)$ defined in the introduction. The following result holds.

Theorem 3.4. *For any $P \in \partial\Omega$,*

$$I(u_{\varepsilon_j}, v_{\varepsilon_j}) \leqslant \varepsilon_j^N [I_\infty(u,v) - \varepsilon_j(N-1)H(P)\gamma + o(\varepsilon_j)] \quad \text{as } \varepsilon_j \to 0,$$

where γ is the positive constant defined in Lemma 3.2 *and (u,v) is given in Proposition* 2.1.

Proof. By construction (cf. [8]),

$$I_j(\widetilde{u}_j, \widetilde{v}_j) \leqslant \sup_{E^- \oplus \mathbb{R}^+(u_j, v_j)} I_j. \tag{3.16}$$

Thus, using Lemmas 3.2 and 3.3 (we also recall from Section 2 the notation $H(P) = \Delta\rho(0)/(N-1)$ for the mean curvature), we deduce

$$I_j(\widetilde{u}_j, \widetilde{v}_j) \leqslant I_\infty(u,v) - \varepsilon_j(N-1)H(P)\gamma + o(\varepsilon_j) \qquad \text{as } \varepsilon_j \to 0.$$

Then the claim easily follows. □

4. Estimating the ground-state levels from below

The estimate of the ground-state level from below is very similar to the one in the previous section and so we only stress the differences. Our next lemma will be used in place of (3.16).

Lemma 4.1. *For any* $e=(e_1,e_2)$ *with* $e_1,e_2\in H^1(\mathbb{R}^N_+)$, *it holds*

$$I_\infty(u,v)\leqslant \sup_{E^-\oplus\mathbb{R}^+e} I_\infty,$$

where $E^-:=\{(\varphi,-\varphi),\varphi\in H^1(\mathbb{R}^N_+)\}$, *provided* $e=(e_1,e_2)\notin E^-$.

Proof. Denote by S the supremum above. It follows from the well-known Benci–Rabinowitz's linking theorem (see e.g. [1]) that I_∞ admits a Palais–Smale sequence $z_n=(u_n,v_n)\in H^1(\mathbb{R}^N_+)\times H^1(\mathbb{R}^N_+)$, such that $\delta\leqslant I_\infty(z_n)\leqslant S+o(1)$ for some $\delta>0$. Denote by $\widetilde{z}_n$ the reflection of z_n with respect to the hyperplane $x_N=0$ and by $\widetilde{I}_\infty$ the energy functional associated to the system in (2.1)–(2.4), but defined in the whole space $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$. Then $\widetilde{z}_n$ is a Palais–Smale sequence for $\widetilde{I}_\infty$ and $\widetilde{I}_\infty(\widetilde{z}_n)=2I_\infty(z_n)$. By standard arguments, up to a subsequence and up to translations, $(\widetilde{z}_n)$ converges weakly to a nonzero critical point $\widetilde{z}$ of $\widetilde{I}_\infty$ and, thanks to Fatou's lemma, $\widetilde{I}_\infty(\widetilde{z})\leqslant 2S$.

Now, since $f(s)=0=g(s)$ for every $s\leqslant 0$, it follows from the maximum principle and from the symmetry result in [3] that, up to a translation, $\widetilde{z}=(\widetilde{z}_1,\widetilde{z}_2)$, where both $\widetilde{z}_1$ and $\widetilde{z}_2$ are positive and radially symmetric with respect to the origin. We denote by z the restriction of $\widetilde{z}$ to $\mathbb{R}^N_+\times\mathbb{R}^N_+$.

As shown in [8] (cf. (3.16) and (3.7)), we have that

$$I_j(\widetilde{u}_j,\widetilde{v}_j)\leqslant I_\infty(z)+\mathrm{o}(1)\leqslant S+\mathrm{o}(1).$$

Taking limits leads to $I_\infty(u,v)\leqslant S$, as claimed. □

Theorem 4.2. *Let* $P_0=\lim_j\widetilde{x}_j$. *We have that*

$$I(u_{\varepsilon_j},v_{\varepsilon_j})\geqslant\varepsilon_j^N[I_\infty(u,v)-\varepsilon_j(N-1)H(P_0)\gamma+o(\varepsilon_j)]\quad\text{as }\varepsilon_j\to 0,$$

where γ *is the positive constant defined in Lemma* 3.2 *and* (u,v) *is given in Proposition* 2.1.

Proof. We will prove that

$$I_j(\widetilde{u}_j,\widetilde{v}_j)\geqslant I_\infty(u,v)-\varepsilon_j(N-1)H(P_0)\gamma+o(\varepsilon_j)\quad\text{as }\varepsilon_j\to 0.$$

Let Φ be a diffeomorphism which straightens a boundary portion near P_0, as in Section 3. For simplicity of notations, we assume that $P_0=0$ and that the unit outward normal of Ω at P_0 is the vector $(0,\dots,0,-1)\in\mathbb{R}^N$. For each j, we also fix a local diffeomorphism Φ^j in such a way that $\Phi^j(0)=\widetilde{x}_j$ and $\det D\Phi^j(0)=1$, and moreover $\Phi^j\to\Phi$ in the C^2 norm, as $j\to\infty$, as well as their inverses $\Psi^j\to\Psi$.

Next, we let

$$\bar{u}_j(x) = \widetilde{u}_j\left(\frac{\Phi^j(\varepsilon_j x) - \widetilde{x}_j}{\varepsilon_j}\right)\chi(\Phi(\varepsilon_j x) - \widetilde{x}_j) \in H^1(\mathbb{R}^N_+), \tag{4.1}$$

where χ is as in (3.1). We observe that $\bar{u}_j$ is indeed well-defined, since $\frac{\Phi^j(\varepsilon_j x) - \widetilde{x}_j}{\varepsilon_j} \in \widetilde{\Omega}_j := \frac{\Omega - \widetilde{x}_j}{\varepsilon_j}$. It is clear that $\frac{\partial \bar{u}_j}{\partial x_N} = 0$ on $\partial\mathbb{R}^N_+$ and that $\lim_{j\to\infty} \bar{u}_j(x) = \lim_{j\to\infty} \widetilde{u}_j(x) = u(x)$, for every x. We define $\bar{v}_j$ in a similar way.

Thanks to Lemma 2.2 we can deduce as in Lemma 3.1 that $\bar{u}_j$ and $\bar{v}_j$ satisfy, over $\mathbb{R}^N_+$,

$$-\Delta\bar{u}_j + \bar{u}_j = g(\bar{v}_j) + \bar{\mu}_j(x), \qquad -\Delta\bar{v}_j + \bar{v}_j = f(\bar{u}_j) + \bar{\nu}_j(x),$$

where $\bar{\mu}_j$ and $\bar{\nu}_j$ are such that

$$\int_{\mathbb{R}^N_+} (\bar{\mu}_j\bar{v}_j + \bar{\nu}_j\bar{u}_j) = \varepsilon_j\Delta\rho(0)\int_{\partial\mathbb{R}^N_+} uv + o(\varepsilon_j) \quad \text{as } j\to\infty. \tag{4.2}$$

It also follows as in Lemma 3.2 that

$$\int_{\mathbb{R}^N_+} (\tfrac{1}{2}f(\bar{u}_j)\bar{u}_j F(\bar{u}_j)) = \int_{\widetilde{\Omega}_j} (\tfrac{1}{2}f(\widetilde{u}_j)\widetilde{u}_j - F(\widetilde{u}_j)) + \varepsilon_j\,\Delta\rho(0)\,\gamma(f) + \mathrm{o}(\varepsilon_j)$$

and, subsequently, thanks to (4.2), that

$$I_\infty(\bar{u}_j, \bar{v}_j) = I_j(\widetilde{u}_j, \widetilde{v}_j) + \varepsilon_j\,\Delta\rho(0)\gamma + \mathrm{o}(\varepsilon_j) \quad \text{as } j\to\infty, \tag{4.3}$$

where γ is the positive constant defined in Lemma 3.2. A part from Lemma 2.2, the main point here is to observe that

$$\det D\Psi^j(\varepsilon_j y + \widetilde{x}_j) = 1 + \varepsilon_j\Delta\rho(0)y_N + o(\varepsilon_j). \tag{4.4}$$

On the other hand, to prove (4.4), let us denote $\beta_j(\varepsilon_j) := \det D\Psi^j(\varepsilon_j y + \widetilde{x}_j)$ and $\beta_0(\varepsilon_j) := \det D\Psi(\varepsilon_j y + P_0)$. Then, using (2.6),

$$\begin{aligned}\det D\Phi(\Psi(\varepsilon_j y + \widetilde{x}_j)) &= 1 - \Delta\rho(0)\Psi_N(\varepsilon_j y + \widetilde{x}_j) + O(|\Psi(\varepsilon_j y + \widetilde{x}_j)|^2)\\ &= 1 - \Delta\rho(0)\varepsilon_j\langle D\Psi_N(\widetilde{x}_j), y\rangle + o(\varepsilon_j)\\ &= 1 - \Delta\rho(0)\varepsilon_j y_N + o(\varepsilon_j),\end{aligned}$$

yielding

$$\beta_0(\varepsilon_j) = (\det D\Phi(\Psi(\varepsilon_j y + \widetilde{x}_j)))^{-1} = 1 + \Delta\rho(0)\varepsilon_j y_N + o(\varepsilon_j);$$

thus

$$\frac{\beta_j(\varepsilon_j)-1}{\varepsilon_j}=\frac{\beta_j(\varepsilon_j)-\beta_j(0)}{\varepsilon_j}=\frac{1}{\varepsilon_j}\int_0^{\varepsilon_j}\beta_j'$$
$$=\frac{1}{\varepsilon_j}\int_0^{\varepsilon_j}(\beta_j'-\beta_0')+\frac{\beta_0(\varepsilon_j)-1}{\varepsilon_j}\to\Delta\rho(0)y_N,$$

since $\beta_j'\to\beta_0'$ uniformly, and (4.4) follows.

Finally, combining Lemma 4.1 and the argument in Lemma 3.3, we see that

$$I_\infty(u,v)\leqslant\sup_{E^-\oplus\mathbb{R}^+(\bar{u}_j,\bar{v}_j)} I_\infty=I_\infty(\bar{u}_j,\bar{v}_j)+o(\varepsilon_j)\quad\text{as } j\to\infty. \tag{4.5}$$

Then from (4.3) and (4.5) we deduce

$$I_j(\widetilde{u}_j,\widetilde{v}_j)\geqslant I_\infty(u,v)-\varepsilon_j(N-1)H(P_0)\gamma+o(\varepsilon_j)\quad\text{as } \varepsilon_j\to 0,$$

as claimed. $\square$

Proof of Theorem 1.2. From Theorems 3.4 and 4.2 it follows that $H(P_0)\geqslant H(P)$, for any $P\in\partial\Omega$. That proves our claim. We also observe that we have shown in Lemma 4.1 that

$$c_\infty := I_\infty(u,v)$$

is the ground-state level for the limit system in $\mathbb{R}^N_+$. In conclusion, we recover for our system an asymptotic formula similar to the one in [4, Eq. (4.13)] for the single equation:

$$I(u_\varepsilon,v_\varepsilon)=\varepsilon^N[c_\infty-\epsilon(N-1)\,H(x_\varepsilon)\,\gamma+\mathrm{o}(\varepsilon)]\qquad\text{as } \varepsilon\to 0.\quad\square$$

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